Journal of Engineering Mathematics **34:** 369–386, 1998.

Green's function formalism extended to systems of applied mechanical differential equations posed on graphs

YU. A. MELNIKOV
Department of Mathematical Sciences, Middle Tennessee State University, Murfreesboro, Tennessee 37132, USA

Received 26 March 1997; accepted in revised form 8 December 1997

Abstract. The Green's function formalism is extended here to multi-point posed boundary-value problems of a special type occurring in some situations in applied mechanics. Problems which reduce to special systems of linear ordinary differential equations are considered. These are formulated on finite weighted graphs in such a way that every equation in the system governs a single unknown function and is defined on a single edge of the graph. The individual equations are put into a system format by means of contact and boundary conditions at the vertices and endpoints of the graph, respectively. Based on such a statement, the notion of the matrix of Green's type is introduced. Two methods are proposed for the analytic construction of such matrices. Illustrative examples from different areas of applied mechanics are presented.

Key words: applied mechanics, piecewise homogeneous media, Green's functions

1. Introduction

Many authors (see, for example, [1–7]) recommend the computational utilization of the Green's function approach to problems of applied mathematical physics. However, the practical use of these functions for actual computations in engineering and science is substantially limited because of a lack of their appropriate representations available in the literature. This situation reflects the fact that the construction of Green's functions for boundary-value problems of mathematical physics is not a routine exercise, even for problems with relatively simple formulations. In an attempt to contribute to this challenging area, we have proposed in [8, 9] and developed in [10, 11] a special technique for constructing Green's functions. This technique was originally suggested for elliptic mixed-type boundary-value problems. It has proven to be productive for a variety of problems in applied continuum mechanics (see [12, pp. 20–59, 79–105, 108–146]).

Classically the Green's function formalism is utilized in situations where governing differential equations have continuous coefficients. However, in [2, 7, 10] attempts were undertaken to extend the Green's function formalism to problems of continuum mechanics, formulated throughout piecewise homogeneous regions, yielding discontinuity of coefficients in the governing differential equations. Later in [11, pp. 15–20], [12, pp. 63–105, 136–146], and [13, 14], an effort has been put forth to implement this formalism for treating the so-called multi-point posed boundary-value problems which model various situations in continuum mechanics for piecewise homogeneous media.

In the earlier works [2], [11, pp. 68–70], and [12, pp. 15–17] an attempt has been undertaken to introduce the notion of the matrix of Green's type. However, the range of application of that notion is limited to the sandwich type of material inhomogeneity. With this in mind, the author's intention in the present study is to introduce the notion of the matrix of Green's type in a different way. The objective is to provide an extension of the Green's function formal-

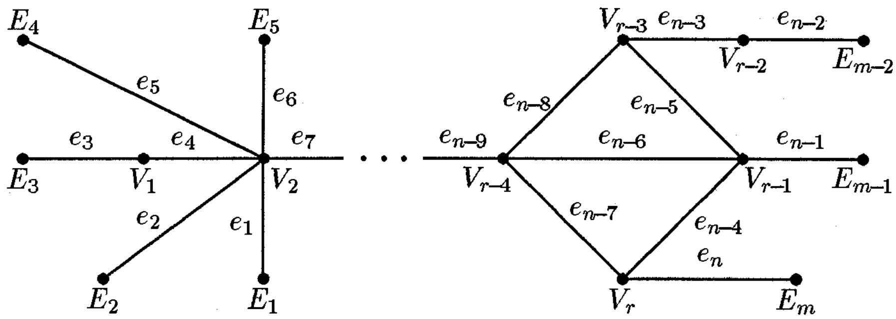

Figure 1. A finite graph R hosting a system of differential equations.

ism to multi-point posed boundary-value problems which model phenomena of continuum mechanics occurring in complex assemblies. For notational convenience, the governing systems of differential equations are set up on finite weighted graphs. This allows a considerably systematic analysis.

2. Definition of the matrix of Green's type

A finite weighted graph R is considered, one containing m endpoints, E_h, $(h = \overline{1,m})$ and r vertices, V_k, $(k = \overline{1,r})$ joined by n edges, $e_i (i = \overline{1,n})$ (see Figure 1). Let positive real numbers l_i, $(i = \overline{1,n})$, representing the lengths of the corresponding edges e_i, be regarded as their weights. Suppose also that every edge e_i of R is occupied with a conductive material (of either thermal or electrical, or any other relevant nature) whose conductivity $p_i(x)$ is a continuously differentiable function of local edge longitudinal coordinate x.

Let $u_i(x)$ represent the unknown function (temperature, electric potential, etc.) defined throughout the edge e_i of R. We will determine the set of these functions by the following set of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(p_i(x)\frac{\mathrm{d}u_i(x)}{\mathrm{d}x}\right) + q_i(x)u_i(x) = -f_i(x), \quad x \in (0, l_i), \ (i = \overline{1,n}). \tag{1}$$

We put these individual equations into a system format by assigning the set of contact conditions

$$u_1(V_k) = \cdots = u_{d_k}(V_k), \qquad \sum_{h=1}^{d_k} p_h(V_k)\frac{\mathrm{d}u_h(V_k)}{\mathrm{d}x} = 0, \quad (k = \overline{1,r}) \tag{2}$$

at each of the vertices V_k, with d_k being their degrees. Notice that for notational convenience, in formulating these conditions, we use a 'local' numbering of the edges incident to the vertex V_k. It can easily be seen that the number of the contact conditions assigned at each vertex equals the degree of the vertex. Clearly, the contact conditions above describe continuity and conservation of energy at every vertex V_k of R. In addition, the boundary conditions

$$\alpha_h\frac{\mathrm{d}u_i(E_h)}{\mathrm{d}x} + \beta_h u_i(E_h) = 0, \quad (h = \overline{1,m}) \tag{3}$$

are imposed at each of the endpoints E_h of R. This implies that $u_i(x)$ in the equations above are defined on the end edges e_i incident to E_h.

From graph theory, it follows that the total number of contact and end conditions posed by Equations (2) and (3) equals $2n$. In this study, we will be focusing on the influence function (matrix) which represents the response of the entire system to a unit energy source acting at an arbitrary point s within an arbitrary edge of R. Notice that the emphasis in this study is on boundary value problems of the type in Equations (1)–(3). However, the results of this work can readily be extended to problems formulated for differential equations of higher order. Later in this paper, we will consider, for example, some formulations from Kirchhoff beam theory, which involve equations of the fourth order on each of the edges of R.

We are now in a position to extend the conventional definition of the Green's function, so as to make it valid for the multi-point posed boundary-value problem of the type in Equations (1–3).

DEFINITION. An $n \times n$ matrix $G(x, s)$, whose entries $g_{ij}(x, s)$ are defined for $x \in e_i$ and $s \in e_j$ on R, is referred to as the *matrix of Green's type* of the homogeneous $(f_i(x) \equiv 0)$ boundary-value problem corresponding to that posed by Equations (1)–(3), if for any fixed value of s, the entries $g_{ij}(x, s)$ have the following properties:

(1) For $x \neq s$, the entries $g_{ii}(x, s)$ of the principal diagonal $(i = j)$ represent continuous functions of x on e_i, have continuous partial derivatives with respect to x up to the second order included, and satisfy the homogeneous $(f_i(x) \equiv 0)$ equations corresponding to those in Equation (1).

(2) At $x = s$, the entries $g_{ii}(x, s)$ of the principal diagonal are continuous functions of x, whereas their partial derivatives of the first order with respect to x are discontinuous functions, provided

$$\frac{\partial g_{ii}(s+0, s)}{\partial x} - \frac{\partial g_{ii}(s-0, s)}{\partial x} = -\frac{1}{p_i(s)}.$$

(3) The peripheral $(i \neq j)$ entries $g_{ij}(x, s)$ of $G(x, s)$ are continuous functions of $x \in e_i$ for any value of $s \in e_j$, have continuous partial derivatives with respect to x up to the second order included, and satisfy the homogeneous equations corresponding to those in Equation (1).

(4) All the entries $g_{ij}(x, s)$ of $G(x, s)$ satisfy the contact and end conditions (which they are involved in) in Equations (2) and (3), in the sense that each of these conditions is satisfied for s belonging to any of the edges e_j, $(j = \overline{1, n})$.

In the discussion that follows, the arguments x and s of the matrix of Green's type are conventionally referred to as the *observation point* and the *source point*, respectively.

3. Existence and uniqueness

The following theorem can be formulated to stipulate the existence and uniqueness of the matrix of Green's type for the homogeneous boundary value problem corresponding to that posed by Equations (1–3).

THEOREM 1. *If the multi-point posed boundary value problem stated by Equations* (1–3) *has a unique solution, then there exists a unique matrix of Green's type* $G(x, s)$ *of the corresponding homogeneous problem* (*that is, the corresonding homogeneous equations subjected to the contact and boundary conditions imposed by Equations* (2) *and* (3)).

If the fundamental solution sets $(u_{i1}(x)$ and $u_{i2}(x)$, $(i = \overline{1, n}))$ of the homogeneous equations corresponding to those in Equation (1) are available, then the proof of this theorem is straightforward. It can readily be accomplished by the standard method based on the defining properties of the matrix of Green's type. Notice that such a proof suggests the procedure for the actual construction of matrices of Green's type for boundary value problems posed on graphs.

Another effective procedure of obtaining matrices of Green's type for homogeneous boundary value problems of the type posed on graphs by Equations (1–3) is based on Lagrange's method of variation of coefficients. To proceed with this method, we introduce a vector-function $\mathbf{U}(\mathbf{x})$ whose components $U_i(x)$, $(i = \overline{1,n})$ are defined in terms of the solutions $u_i(x)$ of Equation (1) as

$$U_i(x) = \begin{cases} u_i(x), & \text{for } x \in e_i, \\ 0, & \text{for } x \in R \backslash e_i. \end{cases}$$

We also introduce a vector-function $\mathbf{F}(\mathbf{x})$ whose components $F_i(x)$ are defined in terms of the right-hand side functions $f_i(x)$ of Equation (1) in the form

$$F_i(x) = \begin{cases} f_i(x), & \text{for } x \in e_i, \\ 0, & \text{for } x \in R \backslash e_i. \end{cases}$$

The following theorem can be proved in the standard way to determine the solution of the boundary value problem posed by Equations (1–3) in terms of the matrix of Green's type of the corresponding homogeneous problem.

THEOREM 2. *If $G(x,s)$ represents the matrix of Green's type of the homogeneous boundary value problem corresponding to that in Equations* (1–3)*, then the solution of the problem posed by Equations* (1–3) *on R can be written as*

$$\mathbf{U}(\mathbf{x}) = \int_R G(x,s)\mathbf{F}(\mathbf{s})\,\mathrm{d}R(s), \quad x \in R, \tag{4}$$

where the integration is carried out over the entire graph R. The converse is also true. That is, if the solution of the problem posed by Equations (1–3) *on R is obtained in the form of the integral in Equation* (4)*, then the kernel $G(x,s)$ of that integral represents the matrix of Green's type for the corresponding homogeneous boundary-value problem.*

Clearly, this theorem suggests that, once the solution of the original problem posed by Equations (1–3) is expressed in terms of the integral in Equation (4), the kernel $G(x,s)$ of such an integral represents the matrix of Green's type of the corresponding homogeneous problem.

A version of the method of variation of coefficients is proposed below to obtain an integral representation of the form in Equation (4) for the solution of the nonhomogeneous boundary-value problem posed by Equations (1–3).

In doing so, we again recall the fundamental solution sets $u_{i1}(x)$ and $u_{i2}(x)$ of the homogeneous equations corresponding to those in Equation (1). The general solution $u_i(x)$ of Equation (1) is sought as follows

$$u_i(x) = D_{i1}(x)u_{i1}(x) + D_{i2}(x)u_{i2}(x), \quad i = \overline{1,n}. \tag{5}$$

Based on that and following the standard procedure of the method of variation of coefficients, we obtain the well-posed systems of linear algebraic equations

$$\begin{pmatrix} u_{i1}(x) & u_{i2}(x) \\ u'_{i1}(x) & u'_{i2}(x) \end{pmatrix} \times \begin{pmatrix} D'_{i1}(x) \\ D'_{i2}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ -f_i(x)/p_i(x) \end{pmatrix}, \quad i = \overline{1,n}$$

in the derivatives of the coefficients $D_{i1}(x)$ and $D_{i2}(x)$ of $u_i(x)$ from Equation (5). From these systems it follows that

$$D'_{i1}(x) = \frac{u_{i2}(x)f_i(x)}{p_i(x)W_i(x)}, \qquad D'_{i2}(x) = \frac{u_{i1}(x)f_i(x)}{p_i(x)W_i(x)}, \quad i = \overline{1,n},$$

where $W_i(x) = u_{i1}(x)u'_{i2}(x) - u_{i2}(x)u'_{i1}(x)$ represent the Wronskians of the fundamental solution sets $u_{i1}(x)$ and $u_{i2}(x)$.

Integration of the derivatives $D'_{i1}(x)$ and $D'_{i2}(x)$ yields

$$D_{i1}(x) = \int_0^x \frac{u_{i2}(s)f_i(s)}{p_i(s)W_i(s)}\,\mathrm{d}s + E_{i1}, \quad i = \overline{1,n},$$

$$D_{i2}(x) = -\int_0^x \frac{u_{i1}(s)f_i(s)}{p_i(s)W_i(s)}\,\mathrm{d}s + E_{i2}, \quad i = \overline{1,n},$$

where E_{i1} and E_{i2} represent undetermined coefficients. By substituting these expressions for $D_{i1}(x)$ and $D_{i2}(x)$ in Equation (5), we can rewrite the latter as

$$\begin{aligned} u_i(x) \;=\; & u_{i1}(x)\int_0^x \frac{u_{i2}(s)f_i(s)}{p_i(s)W_i(s)}\,\mathrm{d}s - u_{i2}(x)\int_0^x \frac{u_{i1}(s)f_i(s)}{p_i(s)W_i(s)}\,\mathrm{d}s \\ & + E_{i1}i_{i1}(x) + E_{i2}u_{i2}(x), \quad i = \overline{1,n}. \end{aligned}$$

Upon combining the integral terms in the equation above, we finally obtain the general solutions of Equation (1) in the form

$$\begin{aligned} u_i(x) \;=\; & \int_0^x \frac{u_{i1}(x)_{i2}(s) - u_{i2}(x)u_{i1}(s)}{p_i(s)W_i(s)} f_i(s)\,\mathrm{d}s \\ & + E_{i1}u_{i1}(x) + E_{i2}u_{i2}(x), \quad x \in [0, l_i], \;\; i = \overline{1,n}. \end{aligned} \tag{6}$$

Upon satisfying the contact and boundary conditions in Equations (2) and (3), whose total number equals $2n$, we can obtain the undetermined coefficients E_{i1} and E_{i2} above, of a total number of also $2n$. This yields a well-posed system of linear algebraic equations which leads finally to the integral representation of the form in Equation (4) for the solution of the problem under consideration. The kernel of that integral represents the matrix of Green's type of the problem.

In the following section, we describe this procedure in detail, while applying it to practical situations occurring in continuum mechanics.

4. Some applications

We will here apply the matrix of Green's type formalism to some problems of continuum mechanics, formulated in media whose properties are discontinuous functions of the space

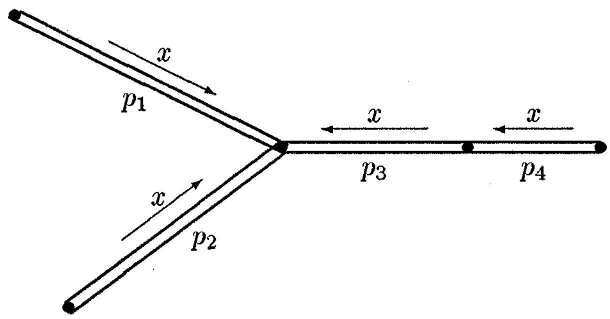

Figure 2. An assembly of heat conductive rods.

variables. The influence matrices will be constructed for steady-state heat conduction in an assembly of rods, for the bending of multi-spaned beams, and for the two-dimensional potential field on a joined shell of revolution.

4.1. STEADY-STATE HEAT CONDUCTION IN AN ASSEMBLY OF RODS

Consider an assembly of rods (see Figure 2), each of which is composed of a homogeneous material whose heat conductivity is p_i.

On the weighted graph associated with the assembly above, we formulate the following multi-point posed boundary-value problem

$$p_i \frac{\mathrm{d}^2 u_i(x)}{\mathrm{d}x^2} = -f_i(x), \quad x \in (0, l_i), \ i = \overline{1,4}, \tag{7}$$

$$u_1(l_1) = u_2(l_2) = u_3(l_3), \tag{8}$$

$$p_1 \frac{\mathrm{d}u_1(l_1)}{\mathrm{d}x} + p_2 \frac{\mathrm{d}u_2(l_2)}{\mathrm{d}x} + p_3 \frac{\mathrm{d}u_3(l_3)}{\mathrm{d}x} = 0, \tag{9}$$

$$u_3(0) = u_4(l_4), \tag{10}$$

$$p_3 \frac{\mathrm{d}u_3(0)}{\mathrm{d}x} - p_4 \frac{\mathrm{d}u_4(l_4)}{\mathrm{d}x} = 0, \tag{11}$$

$$u_1(0) = u_2(0) = u_4(0) = 0, \tag{12}$$

which describes the steady-state heat-conduction phenomenon in the assembly. Here l_i, $(i = \overline{1,4})$ represent the lengths of the rods.

In compliance with the procedure of the method of variation of coefficients described in Section 3, we seek the general solutions of Equation (7) in the form

$$u_{i,g}(x) = D_{i1}(x) + D_{i2}(x)x, \quad i = \overline{1,4},$$

which is ultimately reduced in this case (see Equation (6)) to

$$u_i(x) = \int_0^x \frac{s-x}{p_i} f_i(s)\,\mathrm{d}s + E_{i1} + E_{i2}x, \quad x \in [0, l_i], \ i = \overline{1,4}. \tag{13}$$

We must determine the coefficients E_{i1} and E_{i2}, $(i = \overline{1,4})$ in Equation (13) by satisfying the contact and boundary conditions posed by Equations (8–12). The conditions in Equation (12) yield $E_{11} = E_{21} = E_{41} = 0$. For the rest of the coefficients, we obtain the following well-posed system of linear algebraic equations written as

$$\begin{pmatrix} l_1 & -l_2 & 0 & 0 & 0 \\ l_1 & 0 & -1 & -l_3 & 0 \\ p_1 & p_2 & 0 & p_3 & 0 \\ 0 & 0 & 1 & 0 & -l_4 \\ 0 & 0 & 0 & p_3 & -p_4 \end{pmatrix} \times \begin{pmatrix} E_{12} \\ E_{22} \\ E_{31} \\ E_{32} \\ E_{42} \end{pmatrix} = \begin{pmatrix} A_2 - A_1 \\ A_3 - A_1 \\ B_1 + B_2 + B_3 \\ A_4 \\ -B_4 \end{pmatrix}, \tag{14}$$

where

$$A_i = \int_0^{l_i} \frac{s - l_i}{p_i} f_i(s)\,\mathrm{d}s, \qquad B_i = \int_0^{l_i} f_i(s)\,\mathrm{d}s, \;\; i = \overline{1,4}.$$

For the sake of simplicity, we assume in what follows that the edges of the graph have equal lengths, that is $l_1 = l_2 = l_3 = l_4 = l$. The determinant Δ of the coefficient matrix of the system in Equation (14) is found in this case to be of the form

$$\Delta = l^2[(p_1 + p_2)(p_3 + p_4) + p_3 p_4].$$

When solving the system of Equation (14) and substituting thereupon values of the coefficients E_{i1} and E_{i2} found in Equation (13), we finally obtain

$$\begin{aligned} u_1(x) \; = \; & \int_0^x \frac{s - x}{p_1} f_1(s)\,\mathrm{d}s + \int_0^l \frac{x}{\Delta^* p_1}\{\Delta^* - s[p_2(p_3 + p_4) + p_3 p_4]\} f_1(s)\,\mathrm{d}s \\ & + \int_0^l \frac{xs}{\Delta^*}(p_3 + p_4) f_2(s)\,\mathrm{d}s + \int_0^l \frac{x}{\Delta^*}(lp_3 + sp_4) f_3(s)\,\mathrm{d}s \\ & + \int_0^l \frac{xs}{\Delta^*} p_3 f_4(s)\,\mathrm{d}s, \end{aligned} \tag{15}$$

$$\begin{aligned} u_2(x) \; = \; & \int_0^l \frac{xs}{\Delta^*}(p_3 + p_4) f_1(s)\,\mathrm{d}s + \int_0^x \frac{s - x}{p_2} f_2(s)\,\mathrm{d}s \\ & + \int_0^l \frac{x}{\Delta^* p_2}\{\Delta^* - s[p_1(p_3 + p_4) + p_3 p_4]\} f_2(s)\,\mathrm{d}s \\ & + \int_0^l \frac{x}{\Delta^*}(lp_3 + sp_4) f_3(s)\,\mathrm{d}s + \int_0^l \frac{xs}{\Delta^*} p_3 f_4(s)\,\mathrm{d}s, \end{aligned} \tag{16}$$

$$\begin{aligned} u_3(x) \; = \; & \int_0^l \frac{s}{\Delta^*}(lp_3 + xp_4) f_1(s)\,\mathrm{d}s + \int_0^l \frac{s}{\Delta^*}(lp_3 + xp_4) f_2(s)\,\mathrm{d}s \\ & + \int_0^x \frac{s - x}{p_3} f_3(s)\,\mathrm{d}s + \end{aligned}$$

$$+\int_0^l \frac{1}{\Delta^* p_3}[l(p_1+p_2+p_3)-s(p_1+p_2)](lp_3+xp_4)f_3(s)\,\mathrm{d}s$$
$$+\int_0^l \frac{s}{\Delta^*}[l(p_1+p_2+p_3)-x(p_1+p_2)]f_4(s)\,\mathrm{d}s, \tag{17}$$

$$\begin{aligned} u_4(x) \;=\; & \int_0^l \frac{xs}{\Delta^*}p_3 f_1(s)\,\mathrm{d}s + \int_0^l \frac{xs}{\Delta^*}p_3 f_2(s)\,\mathrm{d}s \\ & + \int_0^l \frac{x}{\Delta^*}[l(p_1+p_2+p_3)-s(p_1+p_2)]f_3(s)\,\mathrm{d}s \\ & + \int_0^x \frac{s-x}{p_4} f_4(s)\,\mathrm{d}s + \int_0^l \frac{x}{\Delta^* p_4}[\Delta^* - sp_3(p_1+p_2)]f_4(s)\,\mathrm{d}s, \end{aligned} \tag{18}$$

where, for notational convenience, we write $\Delta^* = \Delta/l$.

Thus, the solution of the boundary-value problem posed by Equations (7–12) is finally expressed in the form of the integral in Equation (4). This allows the entries $g_{ij}(x,s)$ of the matrix of Green's type $G(x,s)$ of the corresponding homogeneous problem to be read off from the integral representations in Equations (15–18). We exhibit below the entries of the first column of $G(x,s)$

$$g_{11}(x,s) = \frac{1}{\Delta^* p_1}\begin{cases} x\{\Delta^* - s[p_2(p_3+p_4)+p_3p_4]\}, & \text{for } x \leqslant s, \\ s\{\Delta^* - x[p_2(p_3+p_4)+p_3p_4]\}, & \text{for } x \geqslant s, \end{cases}$$

$$g_{21}(x,s) = \frac{xs}{\Delta^*}(p_3+p_4), \qquad g_{31}(x,s) = \frac{s}{\Delta^*}(lp_3+xp_4), \qquad g_{41}(x,s) = \frac{xs}{\Delta^*}p_3,$$

which represent the response of the entire assembly to a unit source acting at the source point s located in the first rod. The rest of the entries of $G(x,s)$ can also be read off from Equations (15–18).

4.2. Bending of multi-spaned beams

The sphere of productive utilization of the methods proposed in this study for the construction of matrices of Green's type is not limited to potential problems. They can also be applied to different phenomena in mechanics. In order to address this issue, we will show in this section how to obtain influence matrices of a unit transverse concentrated force for multi-spaned Kirchhoff beams.

Consider a compound (EI_1 and EI_2) cantilever beam overhanging an intermediate elastic support with a spring constant k^* as shown in Figure 3.

To come forth with the construction procedure, we formulate the following three-point posed boundary-value problem

$$\frac{\mathrm{d}^4 w_1(x)}{\mathrm{d}x^4} = -\frac{q_1(x)}{EI_1} = -f_1(x), \quad x \in (-a, 0), \tag{19}$$

$$\frac{\mathrm{d}^4 w_2(x)}{\mathrm{d}x^4} = -\frac{q_2(x)}{EI_2} = -f_2(x), \quad x \in (0, a), \tag{20}$$

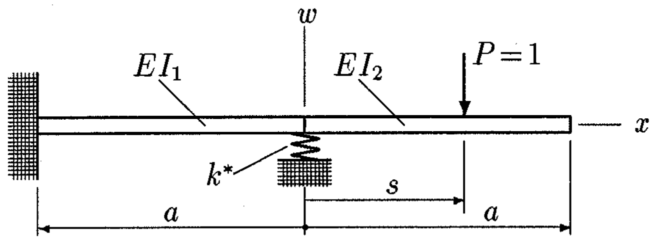

Figure 3. A compound cantilever beam overhanging an elastic support.

$$w_1(-a) = \frac{\mathrm{d}w_1(-a)}{\mathrm{d}x} = 0, \qquad \frac{\mathrm{d}^2 w_2(a)}{\mathrm{d}x^2} = \frac{\mathrm{d}^3 w_2(a)}{\mathrm{d}x^3} = 0, \tag{21}$$

$$w_1(0) = w_2(0), \qquad \frac{\mathrm{d}w_1(0)}{\mathrm{d}x} = \frac{\mathrm{d}w_2(0)}{\mathrm{d}x}, \tag{22}$$

$$EI_1 \frac{\mathrm{d}^2 w_1(0)}{\mathrm{d}x^2} = EI_2 \frac{\mathrm{d}^2 w_2(0)}{\mathrm{d}x^2}, \tag{23}$$

$$EI_1 \frac{\mathrm{d}^3 w_1(0)}{\mathrm{d}x^3} - kw_1(0) = EI_2 \frac{\mathrm{d}^3 w_2(0)}{\mathrm{d}x^3} + kw_2(0), \quad k = 2k^* \tag{24}$$

for the deflection functions $w_1(x)$ and $w_2(x)$ to be determined on the edges $[-a, 0]$ and $[0, a]$ of the corresponding graph, respectively.

Following the technique of the method of variation of coefficients, we present the general solutions of Equations (19) and (20) in the form

$$w_1(x) = \int_{-a}^{x} \frac{(s-x)^3}{6} f_1(s)\,\mathrm{d}s + H_1 + K_1 x + L_1 x^2 + M_1 x^3$$

and

$$w_2(x) = \int_{0}^{x} \frac{(s-x)^3}{6} f_2(s)\,\mathrm{d}s + H_2 + K_2 x + L_2 x^2 + M_2 x^3.$$

In computing values of the undetermined coefficients above, we take advantage of the set of boundary and contact conditions imposed by Equations (21–24). This eventually yields

$$\begin{aligned} w_1(x) \;=\; & \int_{-a}^{x} \frac{(s-x)^3}{6} f_1(s)\,\mathrm{d}s \\ & + \int_{-a}^{0} \frac{(a+x)^2}{6p} \{ks[x(s^2-a^2) - 2a(s^2+ax)] \\ & \qquad +3EI_1[(x+a) - 3(s+a)]\} f_1(s)\,\mathrm{d}s \\ & +\lambda \int_{0}^{a} \frac{(a+x)^2}{2p} \{EI_1[(a+x) - 3(a+s)] - ka^2 xs\} f_2(s)\,\mathrm{d}s \end{aligned} \tag{25}$$

and

$$w_2(x) \;=\; \int_{-a}^{0} \frac{(a+s)^2}{2p} \{EI_1[(a+s) - 3(x+a)] - ka^2 sx\} f_1(s)\,\mathrm{d}s$$

$$+\int_0^x \frac{(s-x)^3}{6} f_2(s)\,\mathrm{d}s + \int_0^a \frac{1}{6p}\{px^2(x-3s) - 3\lambda k a^4 xs$$

$$-3EI_2 a[a(2a+3s)+3x(a+2s)]\} f_2(s)\,\mathrm{d}s, \tag{26}$$

where $p = (2a^3k + 3EI_1)$ and λ represents the ratio EI_2/EI_1.

By virtue of Theorem 2, it follows, from Equations (25) and (26), that the branch $g_{11}^{+}(x,s)$ of the entry $g_{11}(x,s)$ of the matrix of Green's type to the homogeneous problem corresponding to that in Equations (19–24), which is valid for $-a \leqslant x \leqslant s \leqslant 0$, can be written as

$$g_{11}^{+}(x,s) = \frac{(a+x)^2}{6pEI_1}\{ks[x(s^2-a^2) - 2a(s^2+ax)] + 3EI_1[(x+a) - 3(s+a)]\}.$$

Notice that the factor of EI_1 appears in the denominator of $g_{11}^{+}(x,s)$ because the actual right-hand term of Equation (19) is $-q_1(x)/EI_1$.

For $-a \leqslant s \leqslant x \leqslant 0$, we obtain the other branch of $g_{11}(x,s)$ in the form

$$g_{11}^{-}(x,s) = \frac{(a+s)^2}{6pEI_1}\{kx[s(x^2-a^2) - 2a(x^2+as)] + 3EI_1[(s+a) - 3(x+a)]\}.$$

The entry $g_{12}(x,s)$ is defined for $x \in [-a,0]$ and $s \in [0,a]$ and is presented as

$$g_{12}(x,s) = \frac{(a+x)^2}{2pEI_2}\{EI_1[(a+x) - 3(a+s)] - ka^2xs\}.$$

For the entry $g_{21}(x,s)$ defined for $s \in [-a,0]$ and $x \in [0,a]$, we have

$$g_{21}(x,s) = \frac{(a+s)^2}{2pEI_1}\{EI_1[(a+s) - 3(x+a)] - ka^2sx\}.$$

Finally, for $g_{22}^{+}(x,s)$, with both variables x and s belonging to the interval $[0,a]$ and $x \leqslant s$, we obtain

$$g_{22}^{+}(x,s) = \frac{1}{6pEI_2}\{px^2(x-3s) - 3\lambda k a^4 xs - 3EI_2 a[a(2a+3s) + 3x(a+2s)]\},$$

while for $g_{22}^{-}(x,s)$ with $x \geqslant s$, we have

$$g_{22}^{-}(x,s) = \frac{1}{6pEI_2}\{ps^2(s-3x) - 3\lambda k a^4 xs - 3EI_2 a[a(2a+3x) + 3s(a+2x)]\}.$$

Clearly, the matrix of Green's type just obtained can be identified with the influence matrix of a transverse concentrated unit force for the beam under consideration.

For the second example in this section, we determine the response of the compound multi-spaned beam depicted in Figure 4 to a transverse concentrated force P subjected at a point s_0 in the left-hand span.

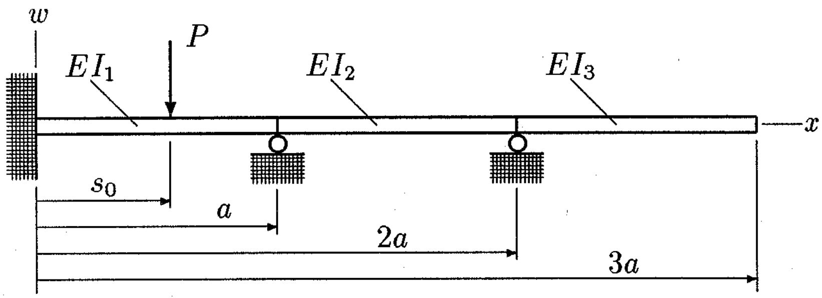

Figure 4. A compound cantilever beam overhanging two supports.

In doing so, we consider the following multi-point posed boundary value problem

$$\frac{\mathrm{d}^4 w_1(x)}{\mathrm{d}x^4} = -\frac{q_1(x)}{EI_1}, \quad x \in (0, a), \tag{27}$$

$$\frac{\mathrm{d}^4 w_2(x)}{\mathrm{d}x^4} = -\frac{q_2(x)}{EI_2}, \quad x \in (a, 2a), \tag{28}$$

$$\frac{\mathrm{d}^4 w_3(x)}{\mathrm{d}x^4} = -\frac{q_3(x)}{EI_3}, \quad x \in (2a, 3a), \tag{29}$$

$$w_1(0) = \frac{\mathrm{d}w_1(0)}{\mathrm{d}x} = 0, \qquad \frac{\mathrm{d}^2 w_3(3a)}{\mathrm{d}x^2} = \frac{\mathrm{d}^3 w_3(3a)}{\mathrm{d}x^3} = 0, \tag{30}$$

$$w_1(a) = w_2(a) = 0, \qquad \frac{\mathrm{d}w_1(a)}{\mathrm{d}x} = \frac{\mathrm{d}w_2(a)}{\mathrm{d}x}, \tag{31}$$

$$EI_1 \frac{\mathrm{d}^2 w_1(a)}{\mathrm{d}x^2} = EI_2 \frac{\mathrm{d}^2 w_2(a)}{\mathrm{d}x^2}, \tag{32}$$

$$w_2(2a) = w_3(2a) = 0, \qquad \frac{\mathrm{d}w_2(2a)}{\mathrm{d}x} = \frac{\mathrm{d}w_3(2a)}{\mathrm{d}x}, \tag{33}$$

$$EI_2 \frac{\mathrm{d}^2 w_2(2a)}{\mathrm{d}x^2} = EI_3 \frac{\mathrm{d}^2 w_3(2a)}{\mathrm{d}x^2} \tag{34}$$

stated on the corresponding three-edge graph.

Following again the standard procedure of the method of variation of coefficients, when solving the boundary-value problem posed by Equations (27–34), we come up with a 3×3 matrix of Green's type $G(x, s)$ (that is, the influence matrix of a unit transverse concentrated force for the beam under consideration). Since the procedure in this case is quite cumbersome, we omit its details and present just the final result.

For the branch $g_{11}^{+}(x, s)$ of $g_{11}(x, s)$, which is valid for both x and s belonging to the interval $[0, a]$ and $0 \leqslant x \leqslant s \leqslant a$, we obtain

$$g_{11}^{+}(x,s)$$
$$=\frac{x^2(a-s)}{6pa^3}\{2[s(2a-s)(x-3a)+2a^2x]+3\lambda_1(a-s)[x(a+s)+s(x-3a)]\},$$

while the other branch $g_{11}^{-}(x,s)$ of $g_{11}(x,s)$, which is valid for $0\leqslant s\leqslant x\leqslant a$, is found to be in the form

$$g_{11}^{-}(x,s)$$
$$=\frac{s^2(a-x)}{6pa^3}\{2[x(2a-x)(s-3a)+2a^2s]+3\lambda_1(a-x)[s(a+x)+x(s-3a)]\},$$

where $p=4EI_1+3EI_2$ and $\lambda_1=EI_2/EI_1$.

For the entry $g_{21}(x,s)$ whose arguments have different domains, namely $x\in[a,2a]$ and $s\in[0,a]$, we obtain

$$g_{21}(x,s)=\frac{1}{2pa^3}s^2(a-s)(x-a)(x-2a)(x-3a)$$

and for $g_{31}(x,s)$, with $x\in[2a,3a]$ and $s\in[0,a]$, we have

$$g_{31}(x,s)=\frac{1}{2pa}s^2(a-s)(2a-x).$$

Clearly, the scalar multiples $Pg_{11}(x,s_0)$, $Pg_{21}(x,s_0)$, and $Pg_{31}(x,s_0)$ represent the deflection functions $w_1(x)$, $w_2(x)$, and $w_3(x)$, defined in the left-hand, the intermediate, and the right-hand spans of the beam, respectively. These deflections are caused by the transverse force of magnitude P concentrated at the point s_0 in the left-hand span. Based on this and recalling the standard relations from beam theory, we can readily obtain explicit expressions for the bending moments and shear forces occurred in this beam, by appropriately differentiating the deflection functions obtained.

Thus, the particular problem posed in the present example is formally solved. Indeed, the response of this beam to the force P applied to an arbitrary point in its left-hand span has already been found. If, however, an external load is also applied to the other span (or spans) of the beam under consideration, then the rest of the entries of the influence matrix ought to be available. With this in mind, we present all these below.

The entry $g_{12}(x,s)$, domains of the arguments for which are $x\in[0,a]$ and $s\in[a,2a]$, is obtained in the form

$$g_{12}(x,s)=\frac{1}{2pa^3}x^2(x-a)(a-s)(2a-s)(3a-s).$$

The branch $g_{22}^{+}(x,s)$ of $g_{22}(x,s)$, with both variables x and s belonging to $[a,2a]$ and $x\leqslant s$, is expressed as

$$g_{22}^{+}(x,s)=\frac{1}{6\lambda_1pa^3}(2a-s)(a-x)\{2(x-a)[s(s-4a)(x-4a)$$
$$+a^2(x-10a)]-3\lambda_1a^2[(s-3a)(s-a)+(x-a)^2]\},$$

while the other branch of $g_{22}(x,s)$, valid for $s \leqslant x$, is found to be in the form

$$g_{22}^{-}(x,s) = \frac{1}{6\lambda_1 pa^3}(2a-x)(a-s)\{2(s-a)[x(x-4a)(s-4a) \\ +a^2(s-10a)] - 3\lambda_1 a^2[(x-3a)(x-a)+(s-a)^2]\}.$$

The entry $g_{32}(x,s)$, with $x \in [2a,3a]$ and $s \in [a,2a]$, is found to be of the following form

$$g_{32}(x,s) = \frac{1}{2\lambda_1 pa}(s-a)(s-2a)(x-2a)[2(a-s)-\lambda_1 s].$$

For $g_{13}(x,s)$, with $x \in [0,a]$ and $s \in [2a,3a]$, we obtain

$$g_{13}(x,s) = \frac{1}{2pa}x^2(a-x)(2a-s).$$

The entry $g_{23}(x,s)$, with $x \in [a,2a]$ and $s \in [2a,3a]$, is expressed as

$$g_{23}(x,s) = \frac{1}{2\lambda_1 pa}(x-a)(x-2a)(s-2a)[2(a-x)-\lambda_1 x].$$

Finally, for the branch $g_{33}^{+}(x,s)$ of $g_{33}(x,s)$, with $2a \leqslant x \leqslant s \leqslant 3a$, we obtain

$$g_{33}^{+}(x,s) = (x-2a)\left\{\frac{1+\lambda_1}{\lambda_1 p}a(2a-s) + \frac{(x-2a)}{6EI_3}[(x+a)+3(a-s)]\right\},$$

while for the other branch $g_{33}^{-}(x,s)$ of $g_{33}(x,s)$, which is defined for $2a \leqslant s \leqslant x \leqslant 3a$, we have

$$g_{33}^{-}(x,s) = (s-2a)\left\{\frac{1+\lambda_1}{\lambda_1 p}a(2a-x) + \frac{(s-2a)}{6EI_3}[(s+a)+3(a-x)]\right\}.$$

Notice that the influence matrix just presented being obtained once, allows us then to compute components of a stress-strain state caused by any reasonable combinations of transverse and bending loads applied to the beam. This highlights one of the key points of the influence function method.

Let, for example, the beam be subjected to continuously distributed transverse loads $q_1(x)$, $q_2(s)$, and $q_3(x)$ applied to its left-hand, intermediate, and the right-hand span, respectively. The influence function method yields then a straightforward formula

$$w_j(x) = \int_0^a g_{j1}(x,s)q_1(s)\,\mathrm{d}s + \int_a^{2a} g_{j2}(x,s)q_2(s)\,\mathrm{d}s \\ + \int_{2a}^{3a} g_{j3}(x,s)q_3(s)\,\mathrm{d}s, \quad (j=1,2,3)$$

for the deflection caused by these loads throughout the entire beam. The subscript j represents here the span number. We can then compute the bending moments and the shear forces in any cross-section of the beam by correspondingly differentiating the expression above with respect to x. If the loading functions $q_1(x)$, $q_2(x)$, and $q_3(x)$ are simple in form, then the integration above can be carried out analytically, otherwise approximate quadrature formulas should be applied for numerical integration.

4.3. Fields of potential on joined shells of revolution

Based on the material of the present study and recalling the technique presented in [12, pp. 79–105], we will derive here the influence matrix of a point source for a two-dimensional field of potential on a joined shell of revolution.

Consider a thin hemispherical shell $\Omega_1 = \{(a, \varphi, \vartheta)\colon 0 \leqslant \varphi \leqslant \frac{1}{2}\pi,\ 0 \leqslant \vartheta < 2\pi\}$, joined to a thin annular plate $\Omega_2 = \{(r, \vartheta)\colon a \leqslant r \leqslant b,\ 0 \leqslant \vartheta < 2\pi\}$, and to another thin hemispherical shell $\Omega_3 = \{(a, \varphi, \vartheta)\colon \frac{1}{2}\pi \leqslant \varphi \leqslant \pi,\ 0 \leqslant \vartheta < 2\pi\}$ to form a 'saturn' type construction. Let each of the fragments be composed of a homogeneous conductive material. The thickness of the construction is assumed to be negligibly small compared to the radius a of the hemispheres and to the width $b - a$ of the annular plate.

Assume the parametrization of the spherical surfaces Ω_1 and Ω_3 as given by geographic coordinates φ and ϑ in the form

$$x = a \sin\varphi \cos\vartheta, \qquad y = a \sin\varphi \sin\vartheta, \qquad z = a \cos\varphi$$

while the annular region Ω_2 is assumed to be parametrized by polar coordinates r and ϑ. For the sake of simplicity in the development that follows, we assume a unit radius $a = 1$ of hemispherical shells Ω_1 and Ω_2.

Clearly, the two-dimensional field of potential in this construction can be determined by the following system of PDEs

$$\frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial v_1(\varphi,\vartheta)}{\partial\varphi}\right) + \frac{1}{\sin^2\varphi}\frac{\partial^2 v_1(\varphi,\vartheta)}{\partial\vartheta^2} = 0, \quad (\varphi,\vartheta) \in \Omega_1, \tag{35}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_2(r,\vartheta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_2(r,\vartheta)}{\partial\vartheta^2} = 0, \quad (r,\vartheta) \in \Omega_2, \tag{36}$$

$$\frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial v_3(\varphi,\vartheta)}{\partial\varphi}\right) + \frac{1}{\sin^2\varphi}\frac{\partial^2 v_3(\varphi,\vartheta)}{\partial\vartheta^2} = 0, \quad (\varphi,\vartheta) \in \Omega_3, \tag{37}$$

subjected to the boundary and contact conditions written as

$$|v_1(0,\vartheta)| < \infty, \qquad v_2(b,\vartheta) = 0, \qquad |v_3(\pi,\vartheta)| < \infty, \tag{38}$$

$$v_1(\tfrac{1}{2}\pi,\vartheta) = v_2(a,\vartheta) = v_3(\tfrac{1}{2}\pi,\vartheta), \tag{39}$$

$$p_1\frac{\partial v_1(\frac{1}{2}\pi,\vartheta)}{\partial\varphi} - p_2\frac{\partial v_2(a,\vartheta)}{\partial r} - p_3\frac{\partial v_3(\frac{1}{2}\pi,\vartheta)}{\partial\varphi} = 0, \tag{40}$$

where p_i, $(i = 1, 2, 3)$ represent the conductivities of the materials of which the fragments Ω_i are composed.

The conditions of boundedness, subjected at the poles $\varphi = 0$ and $\varphi = \pi$ of the hemispheres Ω_1 and Ω_3, respectively, reflect the singularity of the coefficients of Equation (35) and (37) at these points.

Notably, the functions $v_1(\varphi,\vartheta)$, $v_2(\varphi,\vartheta)$, and $v_3(\varphi,\vartheta)$, are 2π-periodic with respect to the variable ϑ. This allows the influence function of a point source for the entire construction

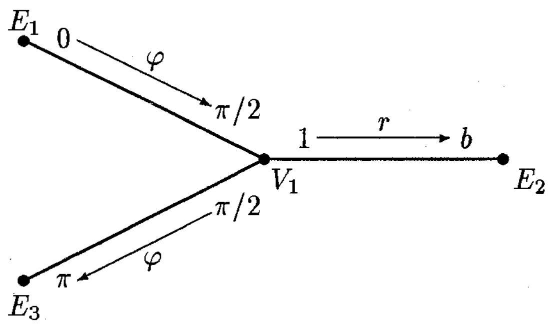

Figure 5. The graph hosting the boundary value problem in Equations (42)–(47).

(which can be identified with the matrix of Green's type $G(x,y;\xi,\eta) = (G_{ij}(x,y;\xi,\eta))$, $(i,j=1,2,3)$ for the boundary value problem in (35)–(40)) to be presented (see [12, pp. 100–105]) by means of the trigonometric series

$$G_{ij}(x,y;\xi,\eta) = \frac{1}{\pi}\left[\tfrac{1}{2}g_{ij}^{0}(x,\xi) + \sum_{n=1}^{\infty} g_{ij}^{n}(x,\xi)\cos(n(y-\eta))\right]. \tag{41}$$

where x and y represent the coordinates of the observation point, while ξ and η represent the coordinates of the source point.

The coefficients $g_{ij}^{n}(x,\xi)$ of the above expansion represent the entries of the matrices of Green's type of the four-point posed boundary-value problems for the set of systems of ordinary differential equations

$$\frac{1}{\sin\varphi}\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\sin\varphi\frac{\mathrm{d}v_{1n}(\varphi)}{\mathrm{d}\varphi}\right) - \frac{n^2}{\sin^2\varphi}v_{1n}(\varphi) = 0, \quad 0<\varphi<\tfrac{1}{2}\pi, \tag{42}$$

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}v_{2n}(r)}{\mathrm{d}r}\right) - \frac{n^2}{r^2}v_{2n}(r) = 0, \quad a<r<b, \tag{43}$$

$$\frac{1}{\sin\varphi}\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\sin\varphi\frac{\mathrm{d}v_{3n}(\varphi)}{\mathrm{d}\varphi}\right) - \frac{n^2}{\sin^2\varphi}v_{3n}(\varphi) = 0, \quad \tfrac{1}{2}\pi<\varphi<\pi, \tag{44}$$

stated on the graph shown in Figure 5.

The system in Equations (42)–(44) is subjected to the boundary conditions

$$|v_{1n}(0)|<\infty, \qquad v_{2n}(b)=0, \qquad |v_{3n}(\pi)|<\infty \tag{45}$$

at the endpoints E_1, E_2, and E_3 of the graph and to the contact conditions

$$v_{1n}(\tfrac{1}{2}\pi) = v_{2n}(a) = v_{3n}(\tfrac{1}{2}\pi), \tag{46}$$

$$p_1\frac{\mathrm{d}v_{1n}(\frac{1}{2}\pi)}{\mathrm{d}\varphi} - p_2\frac{\mathrm{d}v_{2n}(a)}{\mathrm{d}r} - p_3\frac{\mathrm{d}v_{3n}(\frac{1}{2}\pi)}{\mathrm{d}\varphi} = 0 \tag{47}$$

at the graph's vertex V_1.

The statement in Equations (42)–(47) results from the procedure of separation of variables as being applied to the original problem posed by Equations (35)–(40).

The matrices of Green's type for the boundary-value problems in Equations (42)–(47) can be constructed, for example, by means of the method of variation of coefficients. Notably, the case $n = 0$ requires an individual treatment, since the fundamental solution set for this case is different of that for $n \neq 0$. The entries $g_{ij}^0(x,\xi)$ are found in this case to be in the form

$$g_{11}^0(\varphi,\psi) = \begin{cases} \log(b\cot(\frac{1}{2}\psi)), & \varphi \leqslant \psi, \\ \log(b\cot(\frac{1}{2}\varphi)), & \psi \leqslant \varphi \end{cases},$$

$$g_{12}^0(\phi,\rho) = \log(b/\rho), \qquad g_{13}^0(\varphi,\psi) = \log(b),$$

$$g_{21}^0(r,\psi) = \log(b/r), \qquad g_{22}^0(r,\rho) = \begin{cases} \log(b/\rho), & r \leqslant \rho, \\ \log(b/r), & \rho \leqslant r \end{cases},$$

$$g_{23}^0(r,\psi) = \log(b/r),$$

$$g_{31}^0(\varphi,\psi) = \log(b), \qquad g_{32}^0(\varphi,\rho) = \log(b/\rho),$$

$$g_{33}^0(\varphi,\psi) = \begin{cases} \log(b\tan(\frac{1}{2}\varphi)), & \varphi \leqslant \psi, \\ \log(b\tan(\frac{1}{2}\psi)), & \psi \leqslant \varphi. \end{cases}$$

For $n = 1, 2, 3, \dots$, the entries of the matrix of Green's type of the problem posed by (42)–(47) are obtained as follows

$$g_{11}^n(\varphi,\psi) = \frac{1}{2n\Delta}\begin{cases} \tan^n(\frac{1}{2}\varphi)[\Delta\cot^n(\frac{1}{2}\psi) - (b^n + b^{-n})\tan^n(\frac{1}{2}\psi)], & \varphi \leqslant \psi \\ \tan^n(\frac{1}{2}\psi)[\Delta\cot^n(\frac{1}{2}\varphi) - (b^n + b^{-n})\tan^n(\frac{1}{2}\varphi)], & \psi \leqslant \varphi \end{cases},$$

$$g_{12}^n(\varphi,\rho) = \frac{1}{n\Delta}\left[\left(\frac{b}{\rho}\right)^n - \left(\frac{\rho}{b}\right)^n\right]\tan^n(\tfrac{1}{2}\varphi),$$

$$g_{13}^n(\varphi,\psi) = \frac{b^n - b^{-n}}{n\Delta}\tan^n(\tfrac{1}{2}\varphi)\cot^n(\tfrac{1}{2}\psi),$$

$$g_{21}^n(r,\psi) = \frac{1}{n\Delta}\left[\left(\frac{b}{r}\right)^n - \left(\frac{r}{b}\right)^n\right]\tan^n(\tfrac{1}{2}\psi),$$

$$g_{22}^n(r,\rho) = \tfrac{1}{2n\Delta}\begin{cases} (3r^n - r^{-n})[(b/\rho)^n - (\rho/b)^n], & r \leqslant \rho \\ (3\rho^n - \rho^{-n})[(b/r)^n - (r/b)^n], & \rho \leqslant r \end{cases},$$

$$g_{23}^n(r,\psi) = \frac{1}{n\Delta}\left[\left(\frac{b}{r}\right)^n - \left(\frac{r}{b}\right)^n\right]\cot^n(\tfrac{1}{2}\psi),$$

$$g_{31}^n(\varphi,\psi) = \frac{b^n - b^{-n}}{n\Delta}\cot^n(\tfrac{1}{2}\varphi)\tan^n(\tfrac{1}{2}\psi),$$

$$g_{32}^n(\varphi,\rho) = \frac{1}{n\Delta}\left[\left(\frac{b}{\rho}\right)^n - \left(\frac{\rho}{b}\right)^n\right]\cot^n(\tfrac{1}{2}\varphi),$$

$$g_{33}^n(\varphi,\psi) = \frac{1}{2n\Delta}\begin{cases} \cot^n(\frac{1}{2}\psi)[\Delta\tan^n(\frac{1}{2}\varphi) - (b^n + b^{-n})\cot^n(\frac{1}{2}\varphi)], & \varphi \leqslant \psi \\ \cot^n(\frac{1}{2}\varphi)[\Delta\tan^n(\frac{1}{2}\psi) - (b^n + b^{-n})\cot^n(\frac{1}{2}\psi)], & \psi \leqslant \varphi \end{cases},$$

where $\Delta = 3b^n - b^{-n}$.

Partial summation of the series in Equation (41) with the coefficients $g_{ij}^n(x,s)$ just presented can be accomplished in compliance with the method whose detailed description is available in [12, pp. 20–35]. This allows the singular components of the matrix of Green's type $G(x,y;\xi,\eta)$ to be expressed in elementary functions, while its regular components are expressed in the form of uniformly convergent series. Below we find the entries G_{i1} of the first column of this matrix

$$G_{11}(\varphi,\vartheta;\psi,\tau) = -\frac{1}{12\pi}\left\{2\sum_{n=1}^{\infty}\frac{b^n+b^{-n}}{nb^{2n}\Delta}\Phi^n\Psi^n\cos(n(\vartheta-\tau)) - \log\frac{b^2(1-2\Phi\Psi\Theta+\Phi^2\Psi^2)(b^4-2b^2\Phi\Psi\Theta+\Phi^2\Psi^2)}{(\Phi^2-2\Phi\Psi\Theta+\Psi^2)^3}\right\},$$

$$G_{21}(r,\vartheta;\psi,\tau) = \frac{1}{6\pi}\left\{2\sum_{n=1}^{\infty}\frac{1}{nb^{2n}\Delta}\left[\left(\frac{b}{r}\right)^n-\left(\frac{r}{b}\right)^n\right]\Psi^n\cos(n(\vartheta-\tau)) + \log\frac{b^4-2rb^2\Psi\Theta+r^2\Psi^2}{br(r^2-2r\Psi\Theta+\Psi^2)}\right\},$$

$$G_{31}(\varphi,\vartheta;\psi,\tau) = -\frac{1}{6\pi}\left[2\sum_{n=1}^{\infty}\frac{b^n-b^{-n}}{nb^{2n}\Delta}\Phi^{-n}\Psi^n\cos(n(\vartheta-\tau)) + \log\frac{\Phi^2b^4-2b^2\Phi\Psi\Theta+\Psi^2}{b(\Phi^2-2\Phi\Psi\Theta+\Psi^2)}\right],$$

where $\Phi = \tan(\frac{1}{2}\varphi)$, $\Psi = \tan(\frac{1}{2}\psi)$, and $\Theta = \cos(\vartheta-\tau)$.

These entries represent the response of the entire construction to the unit source acting on an arbitrary source point $(\psi,\tau)\in\Omega_1$. For the sake of simplicity, we assumed here that each of the fragments Ω_i of the construction is composed of the same material, that is, $p_1 = p_2 = p_3$.

5. Closing remarks

In the past, the author has undertaken several attempts to adequately extend the traditional Green's function formalism to multi-point posed boundary-value problems of continuum mechanics. However, those attempts were successful only for limited problem classes. It was obvious that different approaches are required to attain a reasonable level of generality. It appears at the moment that the graph-theory framework implemented for this purpose in the present study represents an appropriate basis for approaching the problem. It provides a level of consistency and systematization pertinent to the case.

The notion of the matrix of Green's type is introduced here as one that can be regarded as an adequate extension of the conventional Green's function notion to multipoint posed boundary-value problems of a special kind. Thus, matrices of Green's type, defined in this study, allow a direct implementation of the Green's function formalism to continuum mechanics phenomena occurring in complex assemblies composed of different homogeneous elements.

Acknowledgements

This study has been sponsored in part by the Summer Research Grant of 1997 at Middle Tennessee State University. The author's appreciation goes to Miller Hall, an undergraduate student of the Department of Mathematical Sciences at MTSU, for conducting the computer-algebra-based testing of some of the illustrative examples included in the paper.

References

1. B. A. Boley, A method of the construction of Green's functions. *Q. J. Appl. Math.* 14 (1956) 249–257.
2. S. P. Gavelya, On one method of the construction of Green's matrices for joined shells. *Reports of the Ukrainian Academy of Sciences* A12 (1969) 1107–1111 (in Russian).
3. M. D. Greenberg, *Application of Green's Functions in Science and Engineering.* Englewood Cliffs, N.J.: Prentice-Hall (1971) 141pp.
4. R. Haberman, *Elementary Applied Partial Differential Equations*. Englewood Cliffs, N.J.: Prentice-Hall (1987) 547pp.
5. V. D. Kupradze, *Potential Method in the Theory of Elasticity*. New York: D. Davey (1965) 339pp.
6. G. R. Roach, *Green's Functions*. Boston: Cambridge University Press (1980) 325pp.
7. I. Stakgold, *Green's Functions and Boundary Value Problems*. New York: John Wiley (1979) 638pp.
8. Yu. A. Melnikov, Some applications of the Green's function method in mechanics. *Int. J. Solids and Structures* 13 (1977) 1045–1058.
9. I. M. Dolgova and Yu. A. Melnikov, Construction of Green's functions and matrices for equations and systems of the elliptic type. *Translation from PMM* (*J. Appl. Math. Mech.*) 42 (1978) 740–746.
10. Yu. A. Melnikov and R. D. Krasnikova, *Construction of Green's Functions for Problems of Mathematical Physics*. Dnepropetrovsk: Dnepropetrovsk State University Publishers (1981) 55pp (in Russian).
11. Yu. A. Melnikov, *Green's Functions and Matrices for Equations of Elliptic Type*. Dnepropetrovsk: Dnepropetrovsk State University Publishers (1991) 148pp (in Russian).
12. Yu. A. Melnikov, *Green's Functions in Applied Mechanics*. Boston-Southampton: Computational Mechanics Publications (1995) 267pp.
13. Yu. A. Melnikov, K. L. Shirley and A. J. Worsey, Computing the potential field from a point source on joined surfaces of revolution. In C. A. Brebbia and A. J. Kassab, eds, *Proc. of the Int. Conference BETECH'94*. Orlando: Computational Mechanics Publications (1994) 279–286.
14. Yu. A. Melnikov and K. L. Shirley, Matrices of Green's type for the potential equation on a cylindrical surface joined to a hemisphere. *Appl. Math. and Comp.* 65 (1994) 241–252.